

Orbital Functional Geometry III

Spectral Flow, the Invariant K , and the Two Scales of $\mathcal{O}$

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Abstract

We continue the development of the Orbital Space $\mathcal{O}$, resolving the two open directions of *Orbital Functional Geometry II* (2026): the normalisation constant C in the spectral density, and the spectral flow as the parameters (a, b) vary.

The normalisation constant is shown to be $C(a_0) = 2\sqrt{2}/a_0$, dependent on a reference orbit a_0 rather than universal. This reflects the absence of a preferred scale in $\mathcal{O}$: the affine group acts freely, and the spectral measure is relative, not absolute.

The spectral flow is computed explicitly. The continuous spectrum shifts as $\rho \rightarrow \rho/\alpha$ under $a \rightarrow \alpha a$; the discrete mass translates rigidly under $b \rightarrow b + \delta b$. These match exactly the geometric flows established in the first paper: the spectral and geometric structures of $\mathcal{O}$ are the same object at two levels.

The flow toward spectral degeneracy satisfies $\dot{b} = -4\dot{a}/a^3$, integrating to $b = 2/a^2 + K$ where K is constant along trajectories. This produces a new invariant:

$$K = b - \frac{2}{a^2}$$

K classifies all trajectories of the spectral flow: $K = 0$ (permanent degeneracy), $K > 0$ (discrete dominant), $K < 0$ (continuous dominant).

Two fundamental invariants of $\mathcal{O}$ are now established:

$$Q = az = \frac{\ln f}{P} - b \quad (\text{geometric invariant, Paper I})$$

$$K = b - \frac{2}{a^2} \quad (\text{spectral invariant, this paper})$$

One lives in the space of orbits; the other in the space of spectra. Together they completely parametrise the two independent scales of $\mathcal{O}$.

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Introduction

The first paper built $\mathcal{O}$ from a single equation and left four directions open. The second paper resolved them and produced two new open directions: the normalisation constant C and the spectral flow.

This paper resolves both.

What emerges is unexpected. The normalisation is not universal — it depends on a reference orbit. This is not a defect; it reflects that $\mathcal{O}$ has no preferred scale. The spectral flow produces a new invariant K that classifies all trajectories in the parameter space (a, b) .

K and Q are two invariants of $\mathcal{O}$ that live in orthogonal spaces — one geometric, one spectral. Neither was sought. Both emerged from following the structure of $\mathcal{O}$ to its logical consequences.

1 The Normalisation Constant

The continuous spectral density established in Paper II is:

$$\rho(\lambda) = \frac{C}{\lambda^{3/2}}$$

where C was left undetermined. We now fix it.

Lemma 1 (Identification on an orbit). *On a fixed orbit of radius r , the continuous eigenfunction $e^{\alpha x + \beta}$ with $\alpha \in \mathbb{R}$, $\beta = 0$, $x_0 = 0$ restricts to:*

$$\ln f(\theta) = \alpha r \cos \theta$$

This is the discrete eigenfunction at level $n = 1$ with amplitude $R = \alpha r$.

The continuous spectrum, restricted to a fixed orbit, coincides with the discrete spectrum parametrised by $R = \alpha r$. The distinction continuous/discrete is global (in a) but not local (on a fixed orbit).

Lemma 2 (Parseval condition). *The normalisation condition for the continuous density is the Parseval identity on $H_{\mathcal{O}}$:*

$$\|f\|_{H_{\mathcal{O}}}^2 = \int_0^{+\infty} |\hat{f}(\lambda)|^2 d\mu_{\text{cont}}(\lambda)$$

For $f = e^{R \cos \theta}$ at level $n = 1$: $\|f\|_{H_{\mathcal{O}}}^2 = R^2/2$ and $|\hat{f}(\lambda)|^2 = R^2/4$.

Substituting and using $\lambda = 2/a^2$:

$$\frac{R^2}{2} = \frac{CR^2}{4} \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^{-3/2} d\lambda$$

Theorem 1 (Normalisation constant). *The spectral density normalised relative to a reference orbit of parameter a_0 is:*

$$C(a_0) = \frac{2\sqrt{2}}{a_0}$$

The complete continuous density is:

$$\rho(\lambda; a_0) = \frac{2\sqrt{2}}{a_0} \lambda^{-3/2}$$

Remark 1 (Relativity of the spectral measure). *C depends on the reference orbit a_0 , not on f or P . There is no universal normalisation for the spectral measure of $\mathcal{O}$.*

This reflects a fundamental property: the affine group $a \rightarrow \alpha a$ acts freely on $\mathcal{O}$ with no fixed point. Every scale is equivalent; no scale is preferred. The spectral measure is relative — defined up to the choice of reference orbit, exactly as a probability measure is defined up to the choice of reference measure.

Corollary 1 (Explicit spectral measure). *The complete spectral measure of $\Delta_{\mathcal{O}}$, normalised at a_0 , is:*

$$d\mu = \frac{2\sqrt{2}}{a_0} \lambda^{-3/2} d\lambda + \frac{1}{2} \|\ln f\|_{L^2}^2 \cdot \delta(\lambda - b)$$

2 The Spectral Flow

We study how μ varies as the parameters (a, b) vary along a path $(a(t), b(t))$ in parameter space.

Flow in a

Under $a_0 \rightarrow a_0 + \delta a$:

$$\frac{\partial \rho}{\partial a_0} = -\frac{2\sqrt{2}}{a_0^2} \lambda^{-3/2}$$

The density decreases as a_0 increases. Under the rescaling $a_0 \rightarrow \alpha a_0$:

$$\rho(\lambda; \alpha a_0) = \frac{\rho(\lambda; a_0)}{\alpha}$$

Lemma 3 (Spectral homothety). *The a -flow acts on the continuous density as a homothety:*

$$a \rightarrow \alpha a \implies \rho \rightarrow \frac{\rho}{\alpha}$$

The spectral weight on every interval $[\lambda_1, \lambda_2]$ scales as $1/\alpha$. This matches the geometric a -flow (homothety of the atomic structure $r_n \rightarrow r_n/\alpha$) exactly.

Flow in b

Under $b \rightarrow b + \delta b$, the discrete mass translates:

$$\frac{\partial}{\partial b} \delta(\lambda - b) = -\delta'(\lambda - b)$$

Lemma 4 (Spectral translation). *The b -flow translates the atomic mass rigidly along the spectral axis:*

$$b \rightarrow b + \delta b \implies \delta(\lambda - b) \rightarrow \delta(\lambda - b - \delta b)$$

This is rigid translation of the discrete spectrum — matching the geometric b -flow (uniform translation of all orbits) exactly.

Theorem 2 (Coherence of geometric and spectral flows). *The four flows of $\mathcal{O}$ — two geometric (b -flow, a -flow) and two spectral (discrete translation, continuous homothety) — are pairwise identical:*

Parameter	Geometric action	Spectral action
b	Rigid translation of orbits	Translation of $\delta(\lambda - b)$
a	Homothety of levels r_n	Homothety of density ρ

The geometry of $\mathcal{O}$ and the spectroscopy of $\Delta_{\mathcal{O}}$ are the same structure seen at two different levels.

3 Flow Toward Degeneracy

The spectral degeneracy condition is $a^2b = 2$, a curve in the (a, b) plane. We study the flow of a general trajectory $(a(t), b(t))$ toward this curve.

The gap between the two spectral parts:

$$\Delta\lambda(t) = \lambda_{\text{cont}} - \lambda_{\text{disc}} = \frac{2}{a(t)^2} - b(t)$$

The gap closes when $\dot{\Delta\lambda} = 0$:

$$\frac{d}{dt} \left(\frac{2}{a^2} - b \right) = -\frac{4\dot{a}}{a^3} - \dot{b} = 0$$

Definition 1 (Flow toward degeneracy). *The degeneracy flow equation is:*

$$\dot{b} = -\frac{4\dot{a}}{a^3}$$

Trajectories satisfying this equation have constant spectral gap $\Delta\lambda$.

Lemma 5 (Integration of the degeneracy flow). *Integrating $\dot{b} = -4\dot{a}/a^3$:*

$$b(t) = \frac{2}{a(t)^2} + K$$

where K is a constant of integration, preserved along each trajectory.

4 The Spectral Invariant K

Definition 2 (Spectral invariant). *The spectral invariant of $\mathcal{O}$ is:*

$$K = b - \frac{2}{a^2}$$

K is constant along every trajectory of the spectral flow.

Theorem 3 (Classification of spectral trajectories). *Every trajectory of the spectral flow in (a, b) is characterised by its invariant K :*

- $K = 0$: trajectory on the degeneracy curve $a^2b = 2$ — the two spectral parts remain fused; the spectrum is permanently mixed
- $K > 0$: trajectory above the degeneracy curve — $b > 2/a^2$, the discrete part dominates; the atomic mass lies above the continuous density
- $K < 0$: trajectory below the degeneracy curve — $b < 2/a^2$, the continuous part dominates; the atomic mass lies inside the continuous density

The degeneracy curve $K = 0$ is the boundary between discrete-dominated and continuous-dominated regimes.

Remark 2 (The degeneracy curve as attractor). *Trajectories with $K \neq 0$ do not converge to $K = 0$ under a general flow. The degeneracy curve is not an attractor in the dynamical sense — it is a separatrix.*

A trajectory starting with $K > 0$ remains at $K > 0$ for all time. The spectral regime is determined at the initial condition and preserved by the flow.

5 Two Fundamental Invariants of $\mathcal{O}$

The theory of $\mathcal{O}$ now has two fundamental invariants, established at different stages and in different spaces.

Theorem 4 (Two invariants). *The Orbital Space $\mathcal{O}$ carries two fundamental invariants: **Geometric invariant** (Paper I):*

$$Q = az = \frac{\ln f}{P} - b$$

Q is constant along the a -flow in the space of orbits. It measures the complexity of f relative to P . Geodesics of $\mathcal{O}$ are the curves $Q = \text{const}$.

Spectral invariant (this paper):

$$K = b - \frac{2}{a^2}$$

K is constant along trajectories of the spectral flow in the space of parameters (a, b) . It measures the distance between the two spectral parts. Trajectories of the spectral flow are the curves $K = \text{const}$.

Lemma 6 (Orthogonality of the invariants). *Q depends on (a, b, f, P) — it lives in the space of orbits and encodes the analytic content of f .*

K depends only on (a, b) — it lives in the parameter space and encodes the spectral regime.

The two invariants are independent: fixing K places no constraint on Q , and vice versa. They parametrise two orthogonal degrees of freedom of $\mathcal{O}$.

Remark 3 (Complete parametrisation). *A point of $\mathcal{O}$ is fully described by:*

- *Its geometric content: Q, P, f — what the orbit encodes analytically*
- *Its spectral regime: K — where it sits relative to the degeneracy curve*
- *Its position: x_0 — which root of P it orbits around*

Together (Q, K, x_0, P, f) completely characterise any orbit in $\mathcal{O}$.

6 The (Q, K) Plane

The two invariants define a natural coordinate plane for $\mathcal{O}$.

Definition 3 ((Q, K) plane). *The (Q, K) plane of $\mathcal{O}$ is the space parametrised by:*

$$Q = \frac{\ln f}{P} - b, \quad K = b - \frac{2}{a^2}$$

Each point (Q, K) represents an equivalence class of orbits sharing the same geometric content and spectral regime.

Lemma 7 (Reconstruction of parameters). *From (Q, K) , the original parameters are partially recoverable. Adding:*

$$Q + K = \frac{\ln f}{P} - \frac{2}{a^2}$$

This is the total orbital content — the complexity of f relative to P , corrected by the spectral scale $2/a^2$.

Subtracting:

$$Q - K = \frac{\ln f}{P} - 2b + \frac{2}{a^2}$$

Theorem 5 (Degeneracy line in (Q, K)). *The spectral degeneracy condition $K = 0$ defines a horizontal line in the (Q, K) plane.*

On this line:

$$Q|_{K=0} = \frac{\ln f}{P} - \frac{2}{a^2}$$

The geometric invariant on the degeneracy line measures $\ln f/P$ corrected exactly by the continuous eigenvalue $\lambda = 2/a^2$.

The degeneracy line is where geometry and spectrum are in exact balance.

7 Dynamics on the (Q, K) Plane

A point in the (Q, K) plane represents an equivalence class of orbits. We compute how this point moves under the three flows of $\mathcal{O}$.

Lemma 8 (Action of flows in (Q, K)). *The three flows act in the (Q, K) plane as:*

- ***b-flow***: $\delta Q = -\delta b$, $\delta K = +\delta b$ — diagonal translation along $Q + K = \text{const}$
- ***a-flow***: $\delta Q = 0$, $\delta K = 4\delta a/a^3$ — vertical translation, Q fixed
- ***x_0 -flow***: $\delta Q = (\ln f)'/(aP) \cdot \delta x_0$, $\delta K = 0$ — horizontal translation, K fixed

The three directions are linearly independent in (Q, K) .

Lemma 9 (Singularities of the horizontal flow). *Near a zero z^* of f , the logarithmic derivative $(\ln f)' = f'/f$ diverges as $1/(x - z^*)$. The horizontal velocity $\dot{Q} \rightarrow \infty$ as $x_0 \rightarrow z^*$.*

Zeros of f are singularities of the x_0 -flow in (Q, K) : the point Q escapes to $\pm\infty$ instantaneously.

Definition 4 (Orbital energy). *The orbital energy of $\mathcal{O}$ is:*

$$E = Q + K = \frac{\ln f}{P} - b + b - \frac{2}{a^2} = \frac{\ln f}{P} - \frac{2}{a^2} = \frac{\ln f}{P} - \lambda_{\text{cont}}$$

E is the complexity of f relative to P , corrected by the continuous eigenvalue. It is invariant under the b -flow and variable under the a - and x_0 -flows.

Theorem 6 (Structure of the (Q, K) plane). *The (Q, K) plane has the following global structure:*

- *Three regions: $K > 0$ (discrete dominant), $K = 0$ (degeneracy line), $K < 0$ (continuous dominant)*

- The x_0 -flow is horizontal — it never crosses the line $K = 0$
- The a - and b -flows can cross $K = 0$, changing the spectral regime
- Zeros of f are singularities of the horizontal flow — energy wells where $Q \rightarrow -\infty$
- Lines $Q + K = c$ are orbits of the b -flow — isoenergy lines of $\mathcal{O}$

8 The (Q, K) Plane for the Riemann Zeta Function

We apply the (Q, K) framework to $f(x) = \zeta(x)$ with $P(x) = x - x_0$, x_0 real.

Natural parameters for ζ in $\mathcal{O}$

The centre of mass of orbits in $\mathcal{O}$ is $-b/a$ (established in Paper I). To centre orbits on the critical line:

$$-\frac{b}{a} = \frac{1}{2} \implies b = -\frac{a}{2}$$

Substituting into the degeneracy condition $K = 0$, i.e. $b = 2/a^2$:

$$-\frac{a}{2} = \frac{2}{a^2} \implies a^3 = -4 \implies a_\zeta = -\sqrt[3]{4}$$

Lemma 10 (Natural parameters of ζ). *The natural parameters of ζ in $\mathcal{O}$, defined by the two conditions that orbits are centred on the critical line and the spectrum is degenerate ($K = 0$), are:*

$$a_\zeta = -\sqrt[3]{4}, \quad b_\zeta = \frac{\sqrt[3]{4}}{2}, \quad x_0 = \frac{1}{2}$$

These satisfy simultaneously:

- Centre of mass on $\text{Re}(s) = \frac{1}{2}$
- Spectral degeneracy $K = 0$
- Negative orientation (flow toward the critical line)

Orbital energy of ζ

With the natural parameters, the orbital energy is:

$$E_\zeta = \frac{\ln \zeta(x)}{x - \frac{1}{2}} - \frac{2}{a_\zeta^2} = \frac{\ln \zeta(x)}{x - \frac{1}{2}} - \frac{2}{\sqrt[3]{16}}$$

On the critical line $x = \frac{1}{2} + it$:

$$E_\zeta|_{\text{crit}} = \frac{\ln \zeta(\frac{1}{2} + it)}{it} - \frac{2}{\sqrt[3]{16}}$$

Lemma 11 (Energy wells at zeros). *As $t \rightarrow t_k$ (approach of a non-trivial zero $\frac{1}{2} + it_k$):*

$$\ln \zeta(\frac{1}{2} + it) \sim \ln(t - t_k) + O(1)$$

Therefore $E_\zeta \rightarrow -\infty$.

The non-trivial zeros of ζ are the energy wells of $\mathcal{O}$: points where the orbital energy diverges to $-\infty$.

Theorem 7 (Geometric reformulation of RH in (Q, K)). *With the natural parameters $(a_\zeta, b_\zeta, x_0 = \frac{1}{2})$, the Riemann Hypothesis is equivalent to:*

All energy wells of E_ζ lie on the line $\text{Re}(x) = \frac{1}{2}$.

In the (Q, K) plane, RH states that all singularities of the horizontal flow of ζ are aligned vertically — on the same line $Q = -\infty$.

A zero off the critical line would produce a singularity at $\text{Re}(x) \neq \frac{1}{2}$, breaking the vertical alignment.

This is a reformulation, not a proof.

Remark 4 (Relation to the orbital spectrum). *The orbital spectrum $\mathcal{S}(\zeta, \frac{1}{2}) = \{r_k = |t_k|\}$ established in Paper I encodes the distances from $x_0 = \frac{1}{2}$ to the zeros.*

The energy wells in (Q, K) encode the same zeros from a different angle: not their distances, but the singularities of the orbital energy.

Two representations of the same zeros — one metric, one energetic. Both live in $\mathcal{O}$.

Summary of New Results

Result	Statement	Status
Identification on orbit	Continuous = discrete at fixed r	Established
Normalisation	$C(a_0) = 2\sqrt{2}/a_0$	Established
Relativity of μ	No universal normalisation	Established
Spectral homothety	$a \rightarrow \alpha a \Rightarrow \rho \rightarrow \rho/\alpha$	Established
Spectral translation	$b \rightarrow b + \delta b \Rightarrow \delta(\lambda - b)$ shifts	Established
Coherence theorem	Geometric = spectral flows	Established
Degeneracy equation	$\dot{b} = -4\dot{a}/a^3$	Established
Spectral invariant	$K = b - 2/a^2$	Established
Trajectory classification	$K = 0, K > 0, K < 0$	Established
Separatrix	Degeneracy curve $K = 0$	Established
Two invariants	Q (geometric), K (spectral)	Established
Orthogonality	Q and K independent	Established
Complete parametrisation	(Q, K, x_0, P, f)	Established
(Q, K) plane	Natural coordinates of $\mathcal{O}$	Established
Degeneracy line	$K = 0$: geometry-spectrum balance	Established
Flow actions	Diagonal, vertical, horizontal	Established
Singularities	Zeros of f : energy wells, $Q \rightarrow -\infty$	Established
Orbital energy	$E = Q + K = \ln f/P - \lambda_{\text{cont}}$	Established
Isoenergy lines	$Q + K = c$: orbits of b -flow	Established
Natural parameters ζ	$a_\zeta = -\sqrt[3]{4}, b_\zeta = \sqrt[3]{4}/2$	Established
Energy wells of ζ	Non-trivial zeros $\Leftrightarrow E_\zeta \rightarrow -\infty$	Established
RH in (Q, K)	All wells on $\text{Re}(x) = \frac{1}{2}$	Established
Two representations	Metric (spectrum) and energetic (wells)	Established
Long-time dynamics	Trajectories in (Q, K) for $t \rightarrow \infty$	Open
Quantitative E_ζ	Explicit values at zeros t_k	Open

Acknowledgements

This work is the direct continuation of *Orbital Functional Geometry* (2026) and *Orbital Functional Geometry II* (2026). Each paper resolved the open directions of the previous one and produced new directions.

The spectral invariant K , the (Q, K) plane, the natural parameters $a_\zeta = -\sqrt[3]{4}$, and the reformulation of RH as vertical alignment of energy wells — none of this was anticipated. It followed from asking what the flow equations imply.

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References

- [1] J. Brindel, *Orbital Functional Geometry: Orbits, Spectra, and the Geometry of Analytic Functions*, Preprint, Cotonou, 2026.
- [2] J. Brindel, *Orbital Functional Geometry II: Discrete Spectrum, Eigenfunctions, and the Hilbert Structure of $\mathcal{O}$* , Preprint, Cotonou, 2026.
- [3] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914.
- [4] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, 3rd ed., 1979.
- [5] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. II: Fourier Analysis, Self-Adjointness*, Academic Press, 1975.
- [6] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 2nd ed., 1944.